Note on Professor Tait's "Quaternion Path" to Determinants of the Third Order

by

Rev. Hugh Martin

4. Note on Professor Tait's "Quaternion Path" to Determinants of the Third Order. By the Rev. Hugh Martin, M.A. Communicated by Professor Kelland.

I have read with much interest Professor Tait's "Note on Determinants of the Third Order" in the *Proceedings* of this Session (pp. 59–61), and admire the method of discovering new properties of Determinants. I am not sure, however, that the properties, when discovered, are more difficult of proof by Determinant methods, and I venture to submit the following as simple and elementary:—

The *first* property, namely,

$$\begin{vmatrix} x+x_1 & y+y_1 & z+z_1 \\ x_1+x_2 & y_1+y_2 & z_1+z_2 \\ x_2+x & y_2+y & z_2+z \end{vmatrix} = 2 \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

is true under greater generality, and the Determinant proof is the same as for the special case.

Let

$$\begin{vmatrix} a_{1,1}+a_{1,2}+\,.\,.\,+a_{1,n-2}+a_{1,n-1}, & .\,.\,, & a_{n,1}+a_{n,2}+\,.\,.\,+a_{n,n-2}+a_{n,n-1} \\ a_{1,2}+a_{1,3}+\,.\,.\,+a_{1,n-1}+a_{1,n}, & .\,.\,, & a_{n,2}+a_{n,3}+\,.\,.\,+a_{n,n-1}+a_{n,n} \\ a_{1,3}+a_{1,4}+\,.\,.\,+a_{1,n}+a_{1,1}, & .\,.\,, & a_{n,3}+a_{n,4}+\,.\,.\,+a_{n,n}+a_{n,1} \\ \cdots\cdots\cdots, & .\,.\,, & \cdots\cdots\cdots \\ \cdots\cdots\cdots, & .\,.\,, & \cdots\cdots\cdots \\ \cdots\cdots\cdots, & .\,.\,, & \cdots\cdots\cdots \\ a_{1,n}+a_{1,1}+\,.\,.\,+a_{1,n-3}+a_{1,n-2}, & .\,.\,, & a_{n,n}+a_{n,1}+\,.\,.\,+a_{n,n-3}+a_{n,n-2} \end{vmatrix} = \nabla$$

Add all the rows to form a new first row, namely,

$$(n-1)\,.\,\overline{a_{1,1}+\,.\,.\,+a_{1,n}},\ .\,.\,,\ (n-1)\,.\,\overline{a_{n,1}+\,.\,.\,+a_{n,n}}\ :$$

take the factor $(n-1)$ outside the determinant: subtract the resulting new first row from each of the others:

then add to it the sum of the others: and we have

$$\nabla = (n-1)\begin{vmatrix} a_{1,n} & a_{2,n} & \dots & a_{n,n} \\ -a_{1,1} & -a_{2,1} & \dots & -a_{n,1} \\ -a_{1,2} & -a_{2,2} & \dots & -a_{n,2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ -a_{1,n-1}, & -a_{2,n-1} & \dots & -a_{n,n-1} \end{vmatrix}$$

The negative signs may be removed and compensated for by the factor $(-1)^{n-1}$ placed outside the Determinant, and the first row may be made the last by multiplying again by $(-1)^{n-1}$, which counteracts the former multiplication. Hence

$$\nabla = (n-1)\begin{vmatrix} a_{1,1} & a_{2,1} & \dots & a_{n,1} \\ a_{1,2} & a_{2,2} & \dots & a_{n,2} \\ \dots & \dots & \dots & \dots \\ a_{1,n} & a_{2,n} & \dots & a_{n,n} \end{vmatrix} \qquad \text{Q. E. D.}$$

When $n=3$, this is the quaternion theorem in question.

The *second* formula is the well-known property of the reciprocal, and a simple proof of it will be given below.

The *third* and *fourth* are immediate translations of Quaternion expressions into Determinant forms. And I suppose there is no analytical difficulty in enlarging the number of Sir William R. Hamilton's symbols, i, j, k (though it would introduce the conception of ideal space of more dimensions than three), and thus extending the reach of the "Quaternion path" to Determinants of any order.

The *fifth* formula, in its algebraical expression, would be

$$\begin{vmatrix} \begin{vmatrix} y & z \\ y_1 & z_1 \end{vmatrix} + \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, & \begin{vmatrix} z & x \\ z_1 & x_1 \end{vmatrix} + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, & \&c. \\ \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} + \begin{vmatrix} y_2 & z_2 \\ y & z \end{vmatrix}, & \&c., & \&c. \\ \begin{vmatrix} y_2 & z_2 \\ y & z \end{vmatrix} + \begin{vmatrix} y & z \\ y_1 & z_1 \end{vmatrix}, & \&c., & \&c. \end{vmatrix} = -2\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}^2$$

Now, using capitals for minors we have, by the first property,

$$\begin{vmatrix} X_2+X & Y_2+Y & Z_2+Z \\ X+X_1 & Y+Y_1 & Z+Z_1 \\ X_1+X_2 & Y_1+Y_2 & Z+Z_2 \end{vmatrix} = 2\begin{vmatrix} X & Y & Z \\ X_2 & Y_2 & Z_2 \\ X_1 & Y_1 & Z_1 \end{vmatrix}$$

$$= -2\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = -2\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}^2 \qquad \text{Q. E. D.}$$

This, of course, may be generalised like the *first* formula; and the result is

$$\begin{vmatrix} A_{1,n} + A_{1,1} + A_{1,2} + \,.\, + A_{1,n-2}, & \&c., \&c. \\ A_{1,1} + A_{1,2} + A_{1,3} + \,.\, + A_{1,n-1}, & \&c., \&c. \\ A_{1,2} + A_{1,3} + A_{1,4} + \,.\, + A_{1,n}, & \&c., \&c. \\ \cdots\cdots\cdots & \cdots \\ A_{1,n-1} + A_{1,n} + A_{1,1} + \,.\, + A_{1,n-3}, & \&c., \&c. \end{vmatrix}$$

n/

$$= -(n-1)\begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{1,n} & \cdots & a_{n,n} \end{vmatrix}^{n-1}$$

h

The *sixth* formula is got by a reduplication of the proof of the *fifth*, whether the method used be Quaternion or Determinant.

As to the reciprocal, the following seems quite elementary:—

Let

$$\begin{vmatrix} yz_1 - zy_1 & zx_1 - xz_1 & xy_1 - yx_1 \\ y_1z_2 - z_1y_2 & z_1x_2 - x_2z_1 & x_1y_2 - y_1x_2 \\ y_2z - z_2y & z_2x - xz_2 & x_2y - y_2x \end{vmatrix} = \nabla$$

Multiply the first row by x_2, the second by x, the third by x_1; add all the rows for a new first row; divide the second row by x, and the third by x_1.

$$\begin{vmatrix} \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} & 0 & 0 \\ X & Y & Z \\ X_1 & Y_1 & Z_1 \end{vmatrix} = x_2\nabla$$

Hence

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \begin{vmatrix} \mathrm{Y} & \mathrm{Z} \\ \mathrm{Y}_1 & \mathrm{Z}_1 \end{vmatrix} = x_2 \nabla$$

But it is at once seen that

$$\begin{vmatrix} \mathrm{Y} & \mathrm{Z} \\ \mathrm{Y}_1 & \mathrm{Z}_1 \end{vmatrix} = x_2 \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

Hence

$$\nabla = \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}^2 \qquad \text{Q. E. D.}$$